Combinatorics Solutions

KMHS Varsity Fall 2017

Problems 1-5 were easy and they all had the same answer, 56.

Problem 6. How many groups of two or three people may be formed from a group of 7 people?

Solution. The answer is $\binom{7}{2} + \binom{7}{3} = 56$. Is it a coincidence that $\binom{7}{2} + \binom{7}{3} = \binom{8}{3}$? If we change this problem to choosing from a group of 8 people instead of 7, will the pattern continue? With 8 people, the answer is $\binom{8}{2} + \binom{8}{3} = 84 = \binom{9}{3}$, which follows the same pattern! (Note that we have already solved the problem, but our investigation does not end there. If you happen to see a nice property in a problem, always investigate further.) Thus, we will try to prove that our pattern is general:

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!}.$$

This follows from the definition for binomial coefficients. We already suspect that this complicated expression is $\binom{n}{r}$, so we pull out a factor of $\binom{n}{r}$, leaving a remaining factor which should be 1:

$$= \frac{n!}{r!(n-r)!}\left(\frac{r!(n-r)!}{n(n-r)!(r-1)!} + \frac{r!(n-r)!}{n(n-r-1)!r!}\right).$$

By pulling out that factor, the simplification of the factor on the right becomes easier.

$$= \binom{n}{r}\left(\frac{1}{n}(r+n-r)\right) = \binom{n}{r}.$$

This is called *Pascal's Identity.* I'll let you try to come up with another proof.
Hint: Think about forming a committee. Now imagine you are in the group from which the committee is being formed. Either you are in the committee or aren't.

Problem 7. Find the coefficient of x^5 in the expansion of $(1 + x + x^2 + x^3 + \dots)^4$.

Solution 1 (Calculus). Let $f(x) = (1 + x + x^2 + x^3 + \dots)^4$. Viewing this as a geometric series, we have

$$f(x) = \frac{1}{(1-x)^4}.$$

We recognize this as the third derivative of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$, up to a constant factor. To verify this, we will apply the chain rule to differentiate it 3 times:

$$\frac{d^3}{dx^3}\frac{1}{1-x} = \frac{d^2}{dx^2}\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{2}{(1-x)^3} = \frac{6}{(1-x)^4}.$$

Now we use the two representations of $f(x)$ to find the coefficient of x^5. In other words,

$$(1+x+x^2+x^3+\dots)^4 = \frac{1}{6}\frac{d^3}{dx^3}(1+x+x^2+\dots+x^8+\dots) = \frac{1}{6}(\dots+8\cdot 7\cdot 6x^5+\dots).$$

So the answer is $8\cdot 7 = 56$.

Solution 2 (Stars and Bars).

$$f(x) = (1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots).$$

We choose x^a from the first factor, x^b from the second factor, x^c from the third factor, and x^d from the fourth factor in the above representation, so the product of these terms is $x^{a+b+c+d}$, where $a+b+c+d=5$ and $0 \le a,b,c,d \le 5$. We will represent the number 5 by 5 dots:

$$\bullet\bullet\bullet\bullet\bullet.$$

Now place 3 dividers to separate the dots as so:

$$\bullet|\bullet\bullet\bullet||\bullet.$$

This represents taking $a=1$, $b=3$, $c=0$, and $d=1$. Here is another example:

$$\bullet|\bullet|\bullet|\bullet\bullet$$

represents $a=1$, $b=1$, $c=1$, $d=2$. To find the answer, we need to find the number of ways to place dividers into the configuration. Since there are 8 spaces into which we can place dots and dividers, we can place the 3 dividers into the 8 spaces $\binom{8}{3}=56$ ways (the dots automatically fill in the other spaces). Note that the name *stars and bars* comes from this, although it's more like dots and dividers here.

Problem 8. I'm having a party, and I want to set out five bottles of soft drinks. I shop at a store that only sells four types of soft drinks. How many ways can I choose five bottles of soft drinks for my party?

Solution 1 (Classic Star and Bars). We want to choose a type 1 bottles, b type 2 bottles, c type 3 bottles, and d type 4 bottles, where $a+b+c+d=5$ and $0 \le a,b,c,d \le 5$. It is clear that this problem has a stars and bars setting and is essentially the same problem as above, so the answer is $\binom{8}{3}=56$.

Solution 2. We have five possibilities from each type of soda, namely buying 0, 1, 2, 3, 4, or 5 bottles. The coefficient of x^5 in

$$(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5)$$

is the answer. The exponents of x in each factor are the number of bottles chosen from a particular type. We have 4 factors to represent the 4 different types of bottles. As an example, we can choose $1, x^2, x^2, x$ from the four factors and multiplying them we get x^5. So we want to know the number of ways we can write $a+b+c+d=5$, with a,b,c,d

nonnegative. It is clear that this problem is just a word problem form of problem 7. The answer is $\binom{8}{3} = 56$.

Problem 9. How many three-digit numbers can be formed from the set $\{1,2,3,4,5,6,7,8\}$ which have its digits in strictly decreasing order?

Solution. When we choose the three distinct numbers, the order does not matter. So once we have our three digits, we can rearrange them in any order we like, particularly so that the digits are decreasing $\implies \binom{8}{3} = 56$.

Problem 10. Suppose five girls and three boys are to stand in line. In how many ways can they do this if the boys cannot stand next to each other?

Solution. There is only one configuration based on sex. In particular, since there are more girls than boys, we can make a diagram in which girls are always between boys. We represent each possible spot a boy can be in line with an underscore, resulting in the following diagram.

$$\_G\_G\_G\_G\_G\_$$

There are 6 spots available to place the first boy, 5 remaining spots to place the second boy, and 4 remaining spots to place the third boy (alternatively, we could first choose 3 of the 6 spots and then permute the boys: $3!\binom{6}{3}$). The girls can be permuted $5!$ ways. Therefore, the answer is $6 \cdot 5 \cdot 4 \cdot 5! = 14400$.

Problem 11. Suppose five girls and three boys are to stand in a line. In how many ways can they do this if the boys must stand next to each other?

Solution. Consider the boys as a 3-person unit. We can move the unit along the line in 6 ways:

$$BBBGGGGG, \quad GBBBGGGG \quad GGBBBGGG \quad GGGBBBGG \quad GGGGBBBG$$

$$\text{and} \quad GGGGGBBB.$$

Then we permute the boys $3!$ ways and the girls $5!$ ways $\implies 6 \cdot 3! \cdot 5! = 4320$.

Problem 12. Girls cannot stand next to each other.

Solution. We try the same approach as in problem 10:

$$\_B\_B\_B\_.$$

We can't put anymore underscores since then two girls will be next to each other, so there are no configurations in which this is possible $\implies$ 0.

Problem 14. There must be a girl at each end of the line.

Solution. There are 5 ways to choose a girl for the first end and 4 ways to choose a girl for the other end. We do not care how the girls and boys are arranged with respect to sex in the middle, so there are 6! ways to arrange the middle, for an answer of $5 \cdot 4 \cdot 6! = 14400$.

Problem 16. Two specific girls refuse to stand next to each other.

Solution 1. We use complementary counting by counting the number of ways that the two girls can be together and then subtracting this from the total number of ways to arrange the 8 people. This uses the same idea as problem 11. There are 8! ways to arrange the people, 7 ways to move the 2-girl unit, 2 ways to permute them in the unit, and 6! ways to permute the others, for an answer of $8! - 7 \cdot 2 \cdot 6! = 30240$.

Solution 2. We can force the solution – we know that the girls stand next to each other if and only if there is no one between them. Therefore, we are motivated to construct the following diagram:

$$\_ \bullet \_ \bullet \_ \bullet \_ \bullet \_ \bullet \_ \bullet \_.$$

The underscores show all valid positions to place each girl from the two that refuse to stand next to each other and the dots represent all individuals other than the two girls who refuse to stand next to each other. Since there is always at least one individual separating any two underscores, we know that any choice of underscores will yield a valid arrangement. There are 7 choices for the placement of the first girl, 6 choices for the placement of the second girl, and 6! ways of rearranging the other individuals. Therefore, the number of valid permutations of the individuals is $7 \cdot 6 \cdot 6! = 30240$.

Problem 17. Two specific girls refuse to stand next to each other and one boy refuses to stand on either end of the line.

Solution. Our answer will be of the form (total ways of permuting 8 people)−(ways of having the boy at an end of the line)−(ways of having the two girls together)+(ways of having the two girls together and the boy at an end of the line). Draw a Venn Diagram to see why we need to add that last term. There are 8! ways to permute 8 people. There are $2 \cdot 7!$ ways of having the boy at an end of the line. There are $7 \cdot 2 \cdot 6!$ ways of having the two girls together (same idea as solution 1 above). There are $4 \cdot 6!$ ways of doing both, since we put the boy at the 2 endpoints, arrange the girls in the 2-girl unit 2 ways, arrange the 2-girl unit through the remaining 7 people (we ignore the boy at the endpoint) in 6 ways, and permute the rest in 5! ways $\implies$ $2 \cdot 2 \cdot 6 \cdot 5! = 4 \cdot 6!$. So the answer is

$8! - 2 \cdot 7! - 7 \cdot 2 \cdot 6! + 4 \cdot 6! = 23040$.

Problem 18. Suppose five girls and three boys are to stand in a circle. In how many distinct ways can they do this?

Solution. The problem does not indicate any restrictions other than the circular arrangement, so we can ignore sexes. This is a classic result and the answer is 7!. To see why, we can solve the simpler case of 3 people (since listing all the 7! configurations would be tedious). Here are the configurations:

$A \quad B$	$A \quad C$	$B \quad C$	$B \quad A$	$C \quad A$
C	B	A	C	B

and

$$\begin{matrix} C & & B \\ & A & \end{matrix}.$$

We notice that if we rotate the first configuration counterclockwise, we get

$$\begin{matrix} B & & C \\ & A & \end{matrix}$$

which is the same as the third configuration. By rotating it again, we get the fifth configuration. So these three configurations really are the same. Similarly, 2, 4, and 6 are the same. However, 1 and 2 cannot be rotated to each other, so they are different configurations. Thus, there are only 2 ways to arrange 3 people around a circle. In general, we can fix a person in the arrangement and then permute the others through him. In other words, if person A is fixed, there are $n-1$ people we can place to the right of him, $n-2$ people to the right of this guy, and so on. Thus, the general answer is $(n-1)!$, and the answer to our specific problem is $(8-1) = 7!$.
Note: Assume that the formula is (n–1)! unless the problem specifies that rotations are distinct.